ON THE FUNDUMENTAL INVARIANT OF THE HECKE ALGEBRA $H_n(q)$

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The fundumental invariant of the Hecke algebra $H_n(q)$ is the q-deformed class-sum of transpositions of the symmetric group S_n. Irreducible representations of $H_n(q)$, for generic q, are shown to be completely characterized by the corresponding eigenvalues of C_n alone. For S_n more and more invariants are necessary as n inereases. It is pointed out that the q-deformed classical quadratic Casimir of $SU(N)$ plays an analogous role. It is indicated why and how this should be a general phenomenon associated with q-deformation of classical algebras. Apart from this remarkable conceptual aspect C_n can provide powerful and elegant techniques for computations. This is illustrated by using the sequence C_2, $C_3, \cdots,$ C_n to compute the characters of $H_n(q)$.

This talk will be based on [1] and [2]. Much more complete discussions and references to other authors can be found there. More recent developments can be found in [3].

Let me start by recapitulating certain facts concerning the invariants of the classical symmetric group S_n. The single cycle class-sums $\{[p]_n;\ p = 2,\ 3, \cdots,\ n\}$ belong to the centre. Here $[2]_n$ is the sum of transpositions, $[3]_n$ is that of circular permutation of triplets (each term being a product of transpositions) and so on. Their eigenvalues characterize irreducible representations (irreps.) of S_n corresponding to different standard Young tableaux with n boxes.

Definition. *Content of the box in the i-th row and j-th column of the Young tableau is equal to $(j - i)$.*

The symmetric power sums of these contents gives the eigenvalues $\lambda^{\Gamma}_{[p]_n}$ of $[p]_n$ for Γ. Thus

$$\lambda^{\Gamma}_{[2]_n} = \sum_{(i,\, j)\in\Gamma} (j - i) \tag{1}$$

$$\lambda^{\Gamma}_{[3]_n} = \sum_{(i,\, j)\in\Gamma} (j - i)^2 - \frac{1}{2}n(n-1) \tag{2}$$

[a] *Talk presented at the Nankai workshop, Tianjin, 1995 by A. Chakrabarti.*

and so on [1].

For $n \geq 6$ the eigenvalues of $[2]_n$ show degeneracy. As n inereases higher and higher $[p]_n$'s are needed to uniquely characterize each irreducible representation.

The situation changes dramatically as S_n is q-deformed to $H_n(q)$. The eigenvalues of C_n, the q-deformed $[2]_n$, alone suffice to characterize the irreducible representations for arbitrary n. (Throughout only real, positive i.e. generic q is considered.)

I will now show how this becomes possible. The generators of the $H_n(q)$ satisfy

$$\begin{array}{ll} g_i^2 = (q-1)g_i + q & i = 1,\, 2,\, \cdots,\, n-1 \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & i = 1,\, 2,\, \cdots,\, n-2 \\ g_i g_j = g_j g_i & \text{if } |i-j| \geq 2 \end{array} \tag{3}$$

For $q = 1$ one gets S_n. The fundumental invariant is

$$\begin{aligned} C_n \quad = \quad & g_1 + g_2 + \cdots + g_{n-1} + \frac{1}{q}(g_1 g_2 g_1 + g_2 g_3 g_2 + \cdots + g_{n-2} g_{n-1} g_{n-2}) \\ + \quad & \frac{1}{q^2}(g_1 g_2 g_3 g_2 g_1 + g_2 g_3 g_4 g_3 g_2 + \cdots + g_{n-3} g_{n-2} g_{n-1} g_{n-2} g_{n-3}) \\ + \quad & \cdots \\ + \quad & \frac{1}{q^{n-2}} g_1 g_2 \cdots g_{n-2} g_{n-1} g_{n-2} \cdots g_2 g_1 \end{aligned} \tag{4}$$

For $q = 1$, $g_1 g_2 g_1 = (13)$ and so on and one gets back $[2]_n$. The eigenvalue of the fundumental invariant for the Y-tableau Γ can be shown [1] to be the following q-deformation of (1),

$$\Lambda_n^\Gamma = q \sum_{(i,j) \in \Gamma} \frac{q^{j-i} - 1}{q - 1} = q \sum_{(i,j) \in \Gamma} [j-i]_q. \tag{5}$$

Definition. *q-content of the box in the i-th row and j-th column of the Young tableau is equal to $q\ [j-i]_q$.*

Hence Λ_n^Γ is the sum of the q-contents of the boxes of Γ.

Consider, for $n = 6$, the irreducible representations $[4,\ 1,\ 1]$ and $[3,\ 3]$. The box contents are

0	1	2	3
-1			
-2			

[4, 1, 1]

0	1	2
-1	0	1

[3, 3]

For S_6,

$$\Lambda_6^{[4,\,1,\,1]} = \Lambda_6^{[3,\,3]} = 3 \tag{6}$$

For $H_6(q)$,

$$\begin{aligned} \Lambda_6^{[4,\,1,\,1]}(q) &= q^3 + 2\,q^2 + 3\,q - 2 - \tfrac{1}{q} \\ \Lambda_6^{[3,\,3]}(q) &= q^2 + 3q - 1 \end{aligned} \tag{7}$$

Thus the degeneracy is lifted as q moves away from unity. This is the simplest non-trivial example. For the general case one notes:

(i) The q-contents are constant for boxes on the same diagonal of Γ.

(ii) Developping the q-brackets and regrouping terms

$$\Lambda_n^\Gamma = \sum_{k>0} q^k \pi_k^\Gamma - \sum_{k<0} q^{k+1} \nu_k^\Gamma \tag{8}$$

where

$$\pi_k^\Gamma = \sum_{l \geq k > 0} (\text{number of boxes with content } l)$$

$$\nu_k^\Gamma = \sum_{l \leq k < 0} (\text{number of boxes with content } l)$$

From (i) and (ii) it is not difficult to show [1] that Λ_n^Γ *completely determines* Γ *and hence the irrep.*

Set $q = e^\delta$ $(\neq 1)$ and let

$$\tilde{C}_n \equiv \frac{q-1}{q} C_n \tag{9}$$

then [1],

$$\Lambda_{\tilde{C}_n}^\Gamma = \delta \lambda_{[2]_n}^\Gamma + \frac{\delta^2}{2}(\lambda_{[3]_n}^\Gamma + \frac{1}{2}n(n-1)) + \cdots \tag{10}$$

The eigenvalues of all $[p]_n$ $(p = 2, \cdots,\ n)$ in the coefficients of the above series. This is another way of exhibiting that C_n by itself contains information equivalent to that supplied by all $[p]_n$ for S_n.

Projection operators for irreps. can be constructed in terms of C_n in a straightforward way since there is no degeneracy. When the limit $q \to 1$ is taken correctly higher class-sums of S_n appear automatically as necessary to project out the corresponding irrep. of S_n. This is discussed in detail in [1]. Further interesting uses of projection operators can be found in [3].

In [1] a direct relation was given between the eigenvalues of C_n an those of the Casimir of $SU_q(N)$ (q-deformation of the Casimir quadratic in the Cartan-Weyl generators of $SU(N)$) for an irrep. corresponding to a Y-diagram Γ with

n boxes (and at most $N-1$ rows). It was shown that this Casimir C_2 can be so redefined (denoted then by $\tilde{C}_2$) that the eigenvalue for Γ is just

$$\Lambda^{\Gamma}_{\tilde{C}_2} = \sum_{k=1}^{N-1} q^{2(l_k - k)} \tag{11}$$

where l_k is the number of boxes in the k-th row. This was derived using the Gelfand-Zetlin basis [1]. But (11) is, of course, independent of the choice of such a basis. Since

$$l_k \geq l_{k+1}, \qquad (l_k - k) > (l_{k+1} - k - 1) \tag{12}$$

Hence the indices of q in (11) are strictly monotonically decreasing.

Thus even for a reducible representation arising in a certain context (say some model) if one obtains the matrix of $\tilde{C}_2$ from some source and diagonalizes it the coefficient of each block of unit matrix must be of the form

$$\sum_{k=1}^{N-1} q^{2L_k} \qquad (L_k > L_{k+1}) \tag{13}$$

Now setting

$$l_k = L_k + k, \qquad (k = 1, \cdots, N-1) \tag{14}$$

The Y-diagram Γ is completely determined. *Thus the eigenvalue of a suitably q-deformed quadratic Casimir completely characterizes an irrep. of $SU_q(N)$.* (For $q = 1$ or $SU(N)$ one needs, in general, all the invariants upto order N.)

Setting $q = e^{\delta}$,

$$\sum_k q^{2L_k} = 1 + (2\delta)(\sum_k L_k) + \frac{1}{2!}(2\delta)^2(\sum_k L_k^2) + \cdots \tag{15}$$

One can compare (15) with (10).

The coefficients of higher powers of δ contain informations equivalent to those of higher order Casimirs of $SU(N)$.

I present now, without derivation, the relation between C_n and $\tilde{C}_2$ eigenvalues for a Γ with n boxes [1],

$$\left(\frac{q^2-1}{q^2}\right)^2 \Lambda^{\Gamma}_{C_n(q^2)} + \frac{q^2-1}{q^2} n = \Lambda^{\Gamma}_{\tilde{C}_2} + \frac{q^{2(-N+1)} - 1}{q^2 - 1} \tag{16}$$

for ($\sum l_k = n$ in (11)).

The Hecke and q-deformed unitary algebras are well-known to be closely related. But the aspect presented here is more general. *Thus the q-deformed quadratic Casimirs of the other Lie algebras should play analogous roles.* Our investigation is not complete. Here only $SU_q(N)$ has been studied. However in an accompanying talk [4] the foregoing statement is confirmed for $SO_q(5)$. The discussion at the end of [4] gives an idea of the richness of content of the q-deformed Casimirs.

Apart from such remarkable conceptual aspects, C_n or, even better, the sequence $(C_2, C_3, \cdots, C_n)$ nested in $H_n(q)$ can furnish powerful techniques for various goals. As an example, I will indicate below how they can be used to compute characters. A detailed study can be found in [2]. (Another interesting aspect has been studied in [3]).

For the sequence $H_2(q) \subset H_3(q) \subset \cdots \subset H_n(q)$ one defines the Murphy operators

$$L_2 = C_2, \;\; L_3 = C_3 - C_2, \cdots, \;\; L_n = C_n - C_{n-1} \tag{17}$$

One obtains

$$L_p = \sum_{i=1}^{p-1} q^{1-p+i}(g_i g_{i+1} \cdots g_{p-1} \cdots g_{i+1} g_i) \tag{18}$$

and

$$L_{p+1} = \frac{1}{q} g_p L_p g_p + g_p \tag{19}$$

Basis vectors of an irrep. can be specified by sequences of Y-diagrams (indicating how successive boxes are added)

$$\Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_n \tag{20}$$

The eigenvalue of L_i can be shown to be [2]

$$\{\Gamma_i \backslash \Gamma_{i-1}\}_q \equiv q[k_i - p_i]_q \tag{21}$$

where Γ_i is obtained by adding the box $(k_i,\; p_i)$ to Γ_{i-1}. The eigenvalue is the q-content of the last box added. Also

$$tr(L_i)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} tr(L_i)_{\Gamma_{n-1}} \quad (i = 2,\, 3,\, \cdots,\, n-1.) \tag{22}$$

and

$$tr(L_n)_{\Gamma_n} = \sum_{\Gamma_{n-1} \subset \Gamma_n} |\Gamma_{n-1}|\, \{\Gamma_n \setminus \Gamma_{n-1}\}_q \; . \tag{23}$$

where

$$|\Gamma_{n-1}| = \sum_{\Gamma_{n-2}\subset\Gamma_{n-1}} |\Gamma_{n-2}| = dim\ \Gamma_{n-1} \tag{24}$$

For what follows we will need traces of products of *non-consecutive* Murphy operators only. For such products with

$$\alpha_{i+1} \geq \alpha_i + 2$$

$$tr\left(\prod_{i=1}^{\ell} L_{\alpha_i}\right)_{\Gamma_n} = \sum_{\Gamma_{n-1}\subset\Gamma_n} tr\left(\prod_{i=1}^{\ell} L_{\alpha_i}\right)_{\Gamma_{n-1}} \qquad (\text{for } \alpha_l < n) \tag{25}$$

$$tr\left(\prod_{i=1}^{\ell} L_{\alpha_i}\right)_{\Gamma_n} = \sum_{\Gamma_{n-1}\subset\Gamma_n} \{\Gamma_n \setminus \Gamma_{n-1}\}_q\ tr\left(\prod_{i=1}^{\ell-1} L_{\alpha_i}\right)_{\Gamma_{n-1}} \quad (\text{for } \alpha_l = n) \tag{26}$$

The recursion relations (22) to (26) yield easily the traces of the L's and their non-consecutive products [2]. A symbolic program is easy to set up. Taking the trace of each side of (18) and inverting the relation one obtains [2].

$$tr\Big(g_1 g_2 \cdots g_{k-1}\Big) = \left(\frac{q}{q-1}\right)^{k-2} \sum_{i=0}^{k-2} (-1)^i \binom{k-1}{i} tr(L_{k-i}) \tag{27}$$

Similarly, after multiplying both sides of (18) by L_m (non-consecutive),

$$tr\Big((g_1 g_2 \cdots g_{k-1})L_m\Big) = \left(\frac{q}{q-1}\right)^{k-2} \sum_{i=0}^{k-2} (-1)^i \binom{k-1}{i} tr(L_{k-i}\ L_m)\ . \tag{28}$$

Continuing step-wise, with suitable choices of m at each step, it can be shown [2] that one finally obtains in terms of known traces of the type (25) and (26) traces of the form

$$tr\Big((g_1 g_2 \cdots g_{m_1-1})(g_{m_1+1} \cdots g_{m_2-1}) \cdots (g_{m_j+1} \cdots g_p)(g_k g_{k+1} \cdots g_{p+r} \cdots g_{k+1} g_k)\Big) \tag{29}$$

Here the last factor comes from the last non-consecutive L (L_{p+r-1}). This, in general, has overlapping indices with the preceding factors. At each previous step such overlaps has been assumed to be *reduced* (reexpressed as sums of traces of ordered products the g_i's in ascending order of i) so that they have no overlap but, possibly, *cuts* (at $i = m_1,\ m_2, \cdots,\ m_j$, say). This reduction procedure, to be applied again to (29), will be briefly described below. Let us

however first are exhibit the simplest results, illustrating general properties. One obtains [2]

$$tr(g_i) = tr(L_2), \qquad (i = 1, \cdots,\ n-1) \tag{30}$$

$$tr(g_i g_{i+1}) = \Big(\frac{q}{q-1}\Big)\Big(tr(L_3) - 2tr(L_2)\Big), \qquad (i = 1, \cdots,\ n-2) \tag{31}$$

$$tr(g_i g_{i+2}) = \frac{1}{q-1}\Big(-2q\, tr(L_2) + (q+1)^2 tr(L_3) - (1+q^2) tr(L_4) + (q-1) tr(L_2 L_4)\Big) \tag{32}$$

The equalities of the traces in each example illustrate a fundumental lemma [2]: *The trace of product of any number of disjoint sequences, in any irrep., depends only on the lengths of the component connected sequences.*

Note that in (32), $(g_i g_{i+2})$ being disjoint (i.e. a cut at $i+1$) a non-consecutive product $L_2 L_4$ appears on the right. *Note that the L's on the right do not depend on n (of $H_n(q)$).* This is also a general feature.

When there is no overlap and at most one cut (29) can be reduced relatively easily [2]. Thus

$$\begin{aligned} V_k &\equiv tr\Big((g_1 ... g_{k-1})(g_{k+1} ... g_p)(g_{p+1} ... g_{p+r} ... g_{p+1})\Big) \\ &= \sum_{\ell=0}^{r-1} \binom{r-1}{l} q^l (q-1)^{r-l-1} tr\Big((g_1 ... g_{k-1})(g_{k+1} ... g_{p+r-l})\Big) \end{aligned} \tag{33}$$

(For V_0 with $k = 0$ the first factor is defined to be unity). When there is an overlap we introduce *f-expansions* defined below. (For a full account see Appendix [2]). From,

$$g_i^2 = (q-1) g_i + q$$

one deduces

$$g_i^p = f_p g_i + q\ f_{p-1} \tag{34}$$

where

$$f_p = \frac{q^p - (-1)^p}{q+1}$$

It can be shown that (for overlap$= p - l + 1$)

$$tr\Big((g_1 ... g_p)(g_l ... g_{p+r} ... g_l)\Big) = (q-1) \sum_{r=1}^{p-l+1} q^k f_{2(p-l+1-k)+1} V_k + f_{2(p-l+1)+1} V_0 \tag{35}$$

Thus the f-coefficients are determined only by the length of the overlap. The general case with overlap and multiple cuts is treated in App. [2].

Tables of characters (polynomials in q) are given in [2]. Here let me just summerize the main steps:

(1) Traces of (non-consecutive) products of Murphy operators.

(2) Traces of products of g's with cuts and overlap in terms of (1).

(3) Reduction removing overlaps (f-expansions).

References

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